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AN ANALOGUE OF THE PRIME NUMBER THEOREM FOR CLOSED ORBITS OF SHIFTS OF FINITE TYPE AND THEIR SUSPENSIONS

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ABSTRACT
Following the classical procedure developed by Wiener and Ikehara for the proof of the prime number theorem we find an asymptotic formula for the number of closed orbits of a suspension of a shift of finite type when the suspended flow is topologically weak-mixing and when the suspending function is locally constant.

Introduction

In this note we count the number of closed orbits of a suspension of a shift of finite type and obtain asymptotic formulas by following the Wiener–Ikehara [9] proof of the prime number theorem. A dynamical zeta function, one of many which have been studied in recent years in connection with the foundations of statistical mechanics (cf. [8]), plays the role of Riemann's zeta function in the proof.

The suspending function we consider, a positive function assuming only a finite number of values, is perhaps unnecessarily restrictive. It should be clear that once enough information is gathered about the zeta function of more general functions, our proof should extend to the associated suspensions.

The formula in Corollary 3 occurs in Margulis's study of closed geodesics for compact manifolds of negative curvature [6]. One should also notice Bowen's results for Axiom A flows [3] and [4], [5], for recent work on zeta functions. In this connection it is to be expected that a generalisation of our result will subsume Bowen's approximate asymptotic results.

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Let A be a $k \times k$ irreducible zero-one matrix and define

$$X_A = \left\{ x \in \prod_{n=-\infty}^{\infty} \{1, 2, \cdots, k\} : A(x_n, x_{n+1}) = 1 \right\}.$$

We denote by T the *shift* (of finite type) defined by $(Tx)_n = x_{n+1}$. Throughout will be a continuous function defined on X_A assuming only finitely many value. By considering blocks of symbols in place of symbols, if necessary, there is n loss in generality in assuming, as we do, that f is a function of two variables, i. $f(x) = f(x_0, x_1)$.

The Perron–Frobenius theorem assures us that the matrix $\{\exp f(i,j) \cdot A(i,j$ has a maximum positive eigenvalue β with an associated positive eigenvector r,

$$\sum_j \exp f(i,j) \cdot A(i,j) r_j = \beta r_i.$$

Hence the matrix

$$P(i,j) = \exp f(i,j) \cdot A(i,j) r_j / \beta r_i$$

is stochastic and compatible with $A(i,j)$. We refer to P as the stochastic matr associated with f.

The *pressure* of f is

$$\mathcal{P}(f) = \sup_{\mu} \left(\int f d\mu + h(\mu) \right) = \log \beta$$

where the supremum is taken over all T invariant probabilities μ on X_A. ($h(\mu)$ the entropy of T with respect to μ.) The supremum is attained only for th Markov probability m_P defined by the stochastic matrix P.

Now let us suppose that f is strictly positive. The f *suspension* X_A^f of X_A is th compact space consisting of $\{(x,t): x \in X_A, 0 \leqq t \leqq f(x)\}$ with $(x, f(x))$ and $(Tx,$ identified. The f *suspension flow* T_t^f is the one-parameter group of homeomo phisms of X_A^f defined by $T_t^f(x,s) = (x, s+t)$ for $0 \leqq s \leqq g(x)$ and $0 \leqq s + t$ $g(x)$.

If μ is a T invariant probability then we obtain a T_t^f invariant probabilit locally, by taking a direct product of μ with Lebesgue measure on lines and b dividing the resulting measure by the measure of X_A^f. The resulting probability denoted μ^f. Every T_t^f invariant probability is obtainable from some T invariar probability in this way.

The entropy of T_t^f with respect to μ^f (which by definition is the entropy of T is given by

$$h(\mu^f) = h(\mu) \Big/ \int f d\mu$$

(cf. [1]).

It is not difficult to show that there is a unique $\kappa > 0$ such that $P(-\kappa f) = 0$, and therefore

$$h(m) - \kappa \int f dm = 0$$

for a Markov probability m and

$$h(\mu) - \kappa \int f d\mu < 0$$

for all other T invariant probabilities μ. Hence

$$\kappa = \sup_{\mu} \frac{h(\mu)}{\int f d\mu} = \sup_{\mu^f} h(\mu^f).$$

However, the latter quantity is the *topological entropy* of T_t^f, which we denote by $h(T^f)$. We therefore have

Proposition 1. [8] $\mathcal{P}(-h(T^f)f) = 0$.

We shall be considering the flow T_t^f and our results will depend on whether or not T_t^f is *(topologically) weak-mixing*. T_t^f is weak-mixing if there are no continuous eigenfunctions other than the constants, i.e., there is no $a > 0$ and no continuous function $g \neq 0$ such that

$$g(T_t^f x) = e^{2\pi i a t} g(x) \qquad \text{all } t \in \mathbf{R}. \tag{0.1}$$

In this connection one should note

Proposition 2. [7] (0.1) *has a solution* $g \neq 0$ *for some* $a > 0$ *if and only if there is an integer valued continuous function* M *and a continuous function* θ *assuming only finitely many values such that*

$$af = M + \theta \circ T - \theta. \tag{0.2}$$

If f is a function of two variables and $\tau = \{x, Tx, \cdots, T^{n-1}x\}$ ($T^n x = x$, $n \geqq 1$ least) is a closed orbit of T we define the *weight* of τ with respect to f as

$$\begin{aligned} w(\tau) = w(\tau, f) &= \exp(f(x) + \cdots + f(T^{n-1}x) - n\mathcal{P}(f)) \\ &= P(x_0, x_1) \cdots P(x_{n-1}, x_0) < 1. \end{aligned}$$

The *norm* of τ is defined as $N(\tau) = w(\tau)^{-1}$. If f is also strictly positive and if (0.2) holds then

$$w(\tau, -h(T^f)f) = \exp(-h(T^f)(f(x) + \cdots + f(T^{n-1}x)))$$

is an integral power of $\exp(-h(T^f)/a)$. Conversely if $w(\tau, -h(T^f)f)$ is an integral power of a single number for each closed orbit then the same is true of $\exp(f(x) + \cdots + f(T^{n-1}x))$ from which one can conclude that there exists an integral valued function M of two variables and $a > 0$ such that

$$af(i,j) = M(i,j) + \theta(j) - \theta(i)$$

for some θ. Hence we have

PROPOSITION 3. *If f is positive then T^f_t is not weak-mixing if and only if $w(\tau, -h(T^f)f)$ is an integral power of a single number for all closed orbits τ.*

For a function f of two variables (positive or not) we shall say that f is *exceptional* if $w(\tau, f)$ is an integral power of a single number. We shall say that f is *general* if it is not exceptional. For a function $-h(T^f)f$ with f positive, generality is the same as weak-mixing for T^f_t.

It is not difficult to show that f is exceptional if and only if the stochastic matrix P associated with f has the property that for some 'positive' diagonal matrix $\Delta P \Delta^{-1}$ has powers of a single number α^{-1} for its non-zero entries, $\alpha > 1$, α least.

In the following T is the shift (of finite type) defined by the irreducible 0–1 matrix A and f is a function of two variables. We denote by $\pi_f(y)$ the number of closed T orbits of norm less than or equal to y, and when $f > 0$, we denote by $\pi^f(y)$ the number of closed T^f orbits of length less than or equal to y. $\pi(y)$ denotes the number of closed T orbits τ of period $\lambda(\tau)$ less than or equal to y.

THEOREM.

(a) *Exceptional case*:

$$\pi_f(y) \sim \frac{\log \alpha}{\log y} \sum_{\alpha^n \leqq y} \alpha^n \qquad \textit{as } y \to \infty.$$

(b) *General case*:

$$\pi_f(y) \sim \frac{y}{\log y} \qquad \textit{as } y \to \infty.$$

COROLLARY 1.

(a) *Exceptional case*:

$$\#\{\tau : \lambda(\tau)\mathcal{P}(f) - (f(x) + \cdots + f(T^{\lambda(\tau)-1}x)) \leqq \log y\} \sim \frac{\log \alpha}{\log y} \sum_{\alpha^n \leqq y} \alpha^n \quad \textit{as } y \to \infty.$$

(b) *General case*:

$$\#\{\tau : \lambda(\tau)\mathcal{P}(f)-(f(x)+\cdots+f(T^{\lambda(\tau)-1}x))\leqq \log y\}\sim\frac{y}{\log y} \quad \textit{as } y\to\infty.$$

The sums above are over the points of the closed T orbit τ.

When $f=0$, $\mathcal{P}(f)=h(T)$, the topological entropy, and f is an exceptional case with $\log\alpha = h(T)$. Hence

COROLLARY 2. $\pi(y)\sim(1/y)\Sigma_{n\leqq y}\exp nh(T)$.

COROLLARY 3. *If $f>0$ (a function of two variables) then*:
(a) *If T_t^f is not weak-mixing then*

$$\pi^f(y)\sim\frac{\log\alpha}{h(T^f)y}\sum_{n\leqq yh(T^f)/\log\alpha}\alpha^n.$$

(b) *If T_t^f is weak-mixing then*

$$\pi^f(y)\sim\frac{e^{h(T^f)y}}{h(T^f)y}.$$

Corollaries 1 and 2 follow immediately from the theorem. Substituting $-h(T^f)f$ for f in Corollary 1 and remembering that $\mathcal{P}(-h(T^f)f)=0$ we have

(a) $$\#\{\tau : h(T^f)(f+\cdots+f(T^{\lambda(\tau)-1}x))\leqq y\}\sim\frac{\log\alpha}{y}\sum_{n\leqq(y/\log\alpha)}\alpha^n;$$

(b) $$\#\{\tau : h(T^f)(f+\cdots+f(T^{\lambda(\tau)-1}x))\leqq y\}\sim e^y/y$$

and replacing y by $h(T^f)y$ gives Corollary 3.

The density theorem of prime number theory gives an asymptotic formula for the number $\pi_a(y)$ of primes congruent to a mod m:

$$\pi_a(y)\sim y/(\phi(m)\log y) \qquad \text{as } y\to\infty$$

if $(a,m)=1$ ($\pi_a(y)=0$ otherwise) where ϕ is Euler's function. Whilst there is no additive structure to provide us with congruence classes for the objects studied here (a multiplicative structure can be provided formally) we can view regions of a suspension space as rough substitutes. On the other hand, it is not reasonable to ask whether or not a closed orbit is present in a region of space. The appropriate question concerns the length of time the orbit spends in such a region. With this understanding an approximate analogue of the density version of Dirichlet's theorem can be presented as

COROLLARY 4. *If $f>0$ is a function of two variables and if $U = B \times I \subset \{(x,y)\colon 0 \leqq y \leqq f(x)\}$ where B is closed-open and I is an interval then, with $|I| =$ length of I, we have*

$$\pi_U^f(y) = \#\{\tau\colon \text{sojourn time in } U \leqq y\} = \pi^{|I|\chi_B}(y)$$

with the latter given by Corollary 3 (*case* (a)) *where* $\alpha = e^{|I|}$. ($h(T^{|I|\chi_B})$ *is defined by* $\mathcal{P}(-h(T^{|I|\chi_B})|I|\chi_B) = 0$.)

For a proof we simply note that

$$\pi_U^f(y) = \#\left\{\tau : \sum_{x\in\tau} |I|\chi_B(T^t x) \leqq y\right\}$$

for we may substitute for U the region under the function $|I|\chi_B$.

CONJECTURE.† Let f be strictly positive on X_A and suppose that for some $0<\theta<1$ and some $K>0$, $|f(x)-f(y)| \leqq K\theta^n$ whenever $x_i = y_i$ ($|i| \leqq n$). If T^f is weak-mixing then

$$\zeta(s) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{T^n x = x} \exp - s\left(\sum_{i=0}^{n-1} f(T^i x)\right) h(T^f)$$

is analytic in $\mathcal{R}(s) > 1$ (see [8]) *and* has a continuous extension to $\mathcal{R}(s) = 1$ ($s \neq 1$).

If the conjecture is true then, for such f,

$$\pi^f(y) \sim e^{h(T^f)y}/h(T^f)y \qquad \text{as } y \to \infty,$$

by the Wiener–Ikehara method as illustrated in this paper.

1. The zeta function

In [8] Ruelle considers a very general zeta function given by

$$\zeta(z,f) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x\in \mathrm{Fix}\, T^n} \exp(f(x) + \cdots + f(T^{n-1}x)).$$

Here we define

$$\zeta(s) = \exp \sum_{n=1}^{\infty} \frac{\beta^{-ns}}{n} \sum_{x\in \mathrm{Fix}\, T^n} \exp s(f(x) + \cdots + f(T^{n-1}x)) \tag{1.1}$$

† This, and more, has now been proved by M. Pollicott. A joint article extending the results of this paper to Axiom A flows is in preparation.

for s complex and f, β as defined in the introduction. Evidently

$$\zeta(s)=\exp\sum_{n=1}^{\infty}\frac{1}{n}\sum_{x\in \mathrm{Fix}\, T^n} P(x_0,x_1)^s\cdots P(x_{n-1},x_0)^s$$

where non-zero entries of the matrix P are raised to the power s.

Clearly

$$\zeta(s)=\exp\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Tr}(P^s)^{(n)} \tag{1.2}$$
$$=\frac{1}{\det(I-P^s)},$$

where $(P^s)^{(n)}$ is P^s iterated n times. Hence $\zeta(s)$ converges for $R(s)>1$.

Moreover, it is easy to verify that $\zeta(s)$ has an Euler product representation

$$\zeta(s)=\prod_{\tau}\frac{1}{(1-w(\tau)^s)}=\prod_{\tau}\frac{1}{(1-N(\tau)^{-s})} \tag{1.3}$$

valid for $R(s)>1$.

The meromorphic extension $\zeta(s)=1/\det(I-P^s)$ vanishes nowhere and has a pole at $s=1$ since P is stochastic. Since 1 is a simple eigenvalue of P the pole at $s=1$ is simple. To see this one considers the eigenvalues $\beta_1(s),\beta_2(s),\cdots$ of P^s in a small neighbourhood of $s=1$ and writes $\det(I-P^s)=\Pi_{j=1}^{k}(1-\beta_j(s))$. We may suppose that $\beta_1(1)=1$, that $\beta_1(s)$ is analytic and $|1-\beta_j(s)|>\varepsilon>0$ for $j\neq 1$ and all s in a small neighbourhood of 1 (cf. [2]). Hence if $\det(I-P^s)=(s-1)^2\phi(s)$ where ϕ is analytic then

$$|(s-1)\phi(s)|\geqq \varepsilon^{k-1}\left|\frac{1-\beta_1(s)}{s-1}\right|$$

so that $\beta_1'(1)=0$. However, one can show (cf. [7], [8]) that $\beta_1'(1)$ is the entropy of T, with respect to the Markov measure defined by P, which is certainly not zero. We deduce, therefore, that the zero of $\det(I-P^s)$ at 1 is simple, i.e., the pole of $\zeta(s)$ at $s=1$ is simple.

Let us suppose that $\zeta(s)$ has a pole elsewhere on $R(s)=1$, i.e., that $\det(I-P^s)$ has a zero at $s=1+it_0$ with $|t_0|$ least. Then

$$\sum_k P(j,k)P(j,k)^{it_0}\xi_k=\xi_j \tag{1.4}$$

for some non-zero vector ξ, and

$$\sum_k P(j,k)|\xi_k|\geqq|\xi_j|$$

so that, using the Perron–Frobenius theorem, equality must obtain, with $|\xi_j$ independent of j.

We may suppose $|\xi_j| = 1$ and (1.4) says that a convex combination of points oı the unit circle is a point on the unit circle. This means that

$$P(j,k)^{it_0}\xi_k = \xi_j$$

whenever $P(j,k) \neq 0$. By the definition of the weight of a closed orbit we see tha

$$w(\tau)^{it_0} = 1$$

for all closed orbits. In other words

$$w(\tau) = e^{-2\pi m/t_0} = \alpha^{-m} \qquad (\alpha = e^{2\pi/t_0})$$

for some integer m depending on τ. We therefore have

PROPOSITION 4. *$\zeta(s)$ has a pole on $R(s) = 1$ (other than $s = 1$) if and only if is exceptional in which case $\zeta(s)$ is simply periodic with period it_0 (and we can tak $t_0 > 0$).*

We therefore have two cases to consider:

Exceptional

$\zeta(s)$ is non-vanishing and simply periodic with least period it_0, $t_0 > 0$, and i analytic in $R(s) > 1 - \varepsilon$ $(\varepsilon > 0)$ except for simple poles at $1 + nit_0, n$ $0, \pm 1, \cdots$.

General

$\zeta(s)$ is non-vanishing and analytic in an open neighbourhood of $R(s) \geqq$ except for one simple pole at $s = 1$.

Exceptional case

In this case, using simple periodicity, and noting that $\zeta'(s)/\zeta(s)$ has simpl poles at $1 + nit_0, n = 0, \pm 1, \cdots$ with residue -1 we have

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{-2\pi/t_0}{1 - e^{-(2\pi/t_0)(s-1)}} + \phi(s)$$

where $\phi(s)$ is analytic and simply periodic (period it_0) in $R(s) > 1 - \varepsilon$. Hence i the same region

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{2\pi}{t_0}\sum_{n=0}^{\infty} e^{2\pi n/t_0} \cdot e^{-2\pi ns/t_0} + \phi(s). \tag{1.5}$$

General case

In this case

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{-1}{s-1} + \psi(s) \tag{1.6}$$

in a neighbourhood of $R(s) \geqq 1$ in which $\psi(s)$ is analytic.

At this point it is convenient, for the purpose of computation only, to introduce *fictitious orbits* and an analogue of von Mangoldt's function. By a fictitious orbit we mean a formal product τ' of closed orbits

$$\tau' = \tau_1^{l_1} \cdots \tau_m^{l_m}$$

where $\tau_1, \cdots, \tau_m$ are genuine closed orbits and $l_1, \cdots, l_m$ are positive integers. (Primed symbols will always indicate fictitious orbits. Genuine orbits will have no prime.) For such an object we define

$$N(\tau') = N(\tau_1)^{l_1} \cdots N(\tau_m)^{l_m}$$

and

$$\Lambda(\tau') = \log N(\tau) \qquad \text{if } \tau' = \tau^l$$

for some closed orbit τ, l a positive integer, and

$$\Lambda(\tau') = 0 \qquad \text{otherwise.}$$

By (1.3) we have

$$\begin{aligned} \frac{\zeta'(s)}{\zeta(s)} &= -\sum_\tau \log N(\tau) \cdot \frac{N(\tau)^{-s}}{(1-N(\tau)^{-s})} \\ &= -\sum_\tau \log N(\tau) \sum_{m=1}^{\infty} N(\tau)^{-ms} \qquad (1.7) \\ &= -\sum_{\tau'} \frac{\Lambda(\tau')}{N(\tau')^s} . \end{aligned}$$

These identities are valid in $R(s) > 1$.

Exceptional case

In this case there exists $\alpha = e^{2\pi/t_0}$ such that $N(\tau')$ is a positive integral power of α. Thus

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \alpha^{-ns} \left(\sum_{N(\tau')=\alpha^n} \Lambda(\tau') \right) ,$$

and by (1.5) we conclude

$$\sum_{N(\tau')=\alpha^n} \Lambda(\tau') - \frac{2\pi}{t_0} e^{2\pi n/t_0}$$

converges to zero exponentially fast. (See also [8].) Hence, defining

$$F(y) = \sum_{N(\tau')\leqq y} \Lambda(\tau'),$$

we have

PROPOSITION 5. *In the exceptional case,*

$$F(y) \sim \log\alpha \sum_{\alpha^n \leqq y} \alpha^n \qquad \text{as } y \to \infty.$$

General case

In this case, by (1.7), we have

$$\frac{\zeta'(s)}{\zeta(s)} = -\int_1^\infty y^{-s} dF(y)$$

where (as before) $F(y) = \Sigma_{N(\tau')\leqq y} \Lambda(\tau')$. By (1.6)

$$\int_1^\infty y^{-s} dF(y) = \frac{1}{s-1} - \psi(s)$$

where ψ is analytic in a neighbourhood of $R(s) \geqq 1$. Ikehara's Tauberian theorem (cf. [9]) therefore ensures that

PROPOSITION 6. *For the general case we have*

$$F(y) \sim y \qquad \text{as } y \to \infty.$$

2. Proof of Theorem

In all cases we have

PROPOSITION 7. $\log y \; \pi_f(y) \sim F(y)$ *as* $y \to \infty$.

PROOF.

$$F(y) = \sum_{N(\tau')\leqq y} \Lambda(\tau') = \sum_{\tau} k(\tau) \log N(\tau)$$

where $k = k(\tau)$ is the non-negative integer such that $N(\tau)^k \leqq y < N(\tau)^{k+1}$.

Hence

$$F(y) = \sum_{N(\tau)\leqq y} \left[\frac{\log y}{\log N(\tau)}\right] \log N(\tau)$$

$$\leqq \sum_{N(\tau)\leqq y} \log y = \pi_f(y)\cdot \log y.$$

For an estimate in the other direction we first prove that

$$\pi_f(y)/y^\sigma \to 0 \qquad \text{as } y\to\infty$$

when $\sigma > 1$. To see this note that

$$\zeta(\sigma) = \prod_\tau (1 + w(\tau)^\sigma + w(\tau)^{2\sigma} + \cdots)$$

$$\geqq \prod_{w(\tau)\geqq\varepsilon} (1 + w(\tau)^\sigma + w(\tau)^{2\sigma} + \cdots)$$

$$\geqq \frac{1}{(1-\varepsilon^\sigma)^{\theta(\varepsilon)}}$$

where $\theta(\varepsilon) = \Sigma_{w(\tau)\geqq\varepsilon}\, 1$. Hence

$$\log\zeta(\sigma) \geqq \theta(\varepsilon)\log\frac{1}{(1-\varepsilon^\sigma)} \geqq \varepsilon^\sigma\theta(\varepsilon) \qquad \text{and}$$

$$\varepsilon^{\sigma'}\theta(\varepsilon)\to 0 \qquad \text{as } \varepsilon\to 0 \quad \text{if } \sigma' > \sigma > 1.$$

But this is equivalent to $\pi_f(y)/y^{\sigma'}\to 0$ when $\sigma' > 1$.

Put $x = (y/\log y)^\sigma$ where $0 < \sigma < 1$; then

$$\pi_f(y) = \pi_f(x) + \sum_{x<N(\tau)\leqq y} 1$$

$$\leqq \pi_f(x) + \sum_{N(\tau)\leqq y} \frac{\log N(\tau)}{\log x}$$

$$\leqq \pi_f(x) + \frac{1}{\log x}\sum_{N(\tau')\leqq y} \Lambda(\tau')$$

$$= \pi_f(x) + \frac{F(y)}{\sigma(\log y - \log\log y)}$$

and

$$\frac{\log y\,\pi_f(y)}{F(y)} \leqq \frac{\pi_f(x)\cdot y}{x^{1/\sigma}F(y)} + \frac{1}{\sigma}\,\frac{\log y}{(\log y - \log\log y)}\,.$$

However

$$\frac{\pi_f(x)\cdot y}{x^{1/\sigma}\cdot F(y)}=\frac{\pi_f(x)\log y}{F(y)}=\frac{\pi_f(x)/x^{1+\varepsilon}}{F(y)/(x^{1+\varepsilon}\log y)}.$$

The numerator converges to zero as $x\to\infty$ when $\varepsilon>0$. We show that for $\varepsilon>0$ sufficiently small $F(y)/(x^{1+\varepsilon}\log y)\to\infty$ as $y\to\infty$. This will show that

$$\overline{\lim}\,\frac{\log y\pi_f(y)}{F(y)}\leqq\frac{1}{\sigma}\qquad\text{for all }\sigma<1$$

and the proposition will be proved. However

$$\frac{F(y)}{x^{1+\varepsilon}\log y}=\frac{F(y)(\log y)^{\sigma(1+\varepsilon)}}{y^{\sigma(1+\varepsilon)}\log y}$$

$$\geqq\frac{F(y)}{y^{1-\delta}(\log y)^{\delta}}$$

if ε is chosen so that $\sigma(1+\varepsilon)<1$ and $\delta=1-\sigma(1+\varepsilon)>0$. Moreover, by Proposition 5 for the exceptional case and Proposition 6 for the general case $F(y)/y$ is bounded below, so $F(y)/y^{1-\delta}(\log y)^{\delta}\to\infty$ as $y\to\infty$. This completes the proof.

Part (a) of the Theorem is proved by combining Propositions 5 and 7. Part (b) of the Theorem is proved by combining Propositions 6 and 7.

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